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A New Preconditioned Gauss-Seidel Method faster than the Classical Gauss-Seidel Method

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ABSTRACT

In this paper, we present a new preconditioned Gauss-Seidel (GS) method for solving the linear system $Ax = b$ and compared the proposed method with the standard Gauss-Seidel method. We produced some comparison theorems to show the efficiency of the proposed method. Numerical examples are also being studied to know the rate of convergence, memory requirements and time to converge. Finally, the numerical experiments show that the proposed method is better than the standard Gauss-Seidel method, the preconditioned GS methods with the preconditioners such as $(I + C)$ [6], $(I + P_1)$ [8] and $(I + P_2)$ [8].

Keywords: Preconditioning, Convergence, Z-matrix, Spectral radius, GS method.

INTRODUCTION

The use of direct method for solving the linear system $Ax = b \cdots\cdots (1)$ is often impractical from the computational point of view, when the coefficient matrix A is large and sparse. For example solving a linear system with 20 equations and 20 unknowns by Cramer's rule(direct method) using the usual definition of determinant, would require more than a million of years even on a fast computer. Due to this when A is large and sparse, iterative methods are very much essential for solving the linear system (1). Moreover, when the condition number of the coefficient matrix is large then the iterative methods converge very slowly. To minimize the condition number of the coefficient matrix, preconditioning techniques are being often used. In this procedure we shall multiply the linear system (1) by a suitable matrix say P (called the preconditioner) i.e. $PAx = Pb$ and then solve the preconditioning system by an iterative method. In fact, the smaller condition number of the coefficient matrix of a linear system makes the faster rate of convergence of the iteration process. A preconditioner P of a matrix A is a matrix such that $P^{-1}A$ has a smaller condition number than A. It is sometimes common to call $T = P^{-1}$, the preconditioner rather than P, since P itself is rarely explicitely available. Moreover, in order to conclude this paper we shall be looking for the spectral radii of the iteration matrices of the two methods namely classical GS and the proposed method because it is obvious from the iteration convergence theorem that for the convergence of any iterative method, the spectral radius of the iteration matrix has to be less than 1. At last, the smaller spectral radius of the iteration matrix, the faster is the rate of convergence of the iterative method.

For our convenience, we consider the diagonal elements of A are unity and splits A as $A = I - L - U$; where I, L, U are the identity, strictly lower and strictly upper triangular matrices respectively.

Preliminaries

Definition 2.1. The splitting $A = M - N$ is termed as GS splitting if $M = D - L,\ N = U$; where D is a diagonal matrix and $L,\ U$ are strictly lower and strictly upper triangular matrices of A. If $M^{-1} = (D - L)^{-1} \geq 0$ and $N = U \geq 0$, then it is said that A is the GS convergent splitting and this splitting is called convergent when $\rho\left(M^{-1}N\right) < 1$.

Definition 2.2. The splitting $A = M - N$ is an M-splitting if M is nonsingular M-matrix and N is a nonnegative matrix.

Definition 2.3. The GS iteration matrix for the system $Ax = b$ $i.e. (I - L - U)x = b$ is $T = (I - L)^{-1}U$ and the GS method is convergent when $\rho(T) < 1$.

Definition 2.4. A strictly upper triangular matrix is a square matrix whose all the elements in the principal diagonal and below the principal diagonal are zeroes.

Definition 2.5. A strictly lower triangular matrix is a square matrix in which all the elements in the leading diagonal and above the leading diagonal are zeroes.

Definition 2.6. A Z-matrix $A = (a_{ij})_{nxn}$ is a square matrix such that $a_{ij} \leq 0 \ for \ i \neq j.$

Definition 2.7. A Z-matrix A is called an M-matrix if

(a) all real eigenvalues of A are positive.
(b) the real part of any eigenvalue of A is positive.
(c) the diagonal entries of A are positive.
(d) A can be expressed in the form $A = kI - B$, where $B \geq 0 \ and \ \rho(B) \leq k.$

Definition 2.8. A square matrix A is said to be irreducible if the directed graph $D(A)$ of A is strongly connected, otherwise A is reducible.

Lemma 2.9. [10] Let A be a nonnegative nxn nonzero matrix, then

(1) $\rho(A)$, the spectral radius of A, is an eigenvalue;
(2) A has a nonnegative eigenvector corresponding to $\rho(A)$;
(3) $\rho(A)$ is a simple eigenvalue of A;
(4) $\rho(A)$ increases when any entry of A increases.

Proposed method

We consider the preconditioner as $P = I + S_3$ in order to accelerate the convergence rate of the GS method for solving the linear system $Ax = b$; where

$$S_3 = (\ 0 \ \ 0 \ \ \cdots \ \ 0 \ \ -a_{1n} \ 0 \ \ \vdots \ \ 0 \ \ 0 \ \vdots \ 0 \ \ \cdots \ \ddots \ \cdots \ \ 0 \ \vdots \ 0 \ \ -a_{2n} \ \vdots \ -a_{n-1,n} \ \ 0 \ \ 0 \ \ \cdots \ \ 0 \ \ 0 \ )$$

Now, $$Ax = b$$
Or, $$PAx = Pb$$
Or, $$(I + S_3)(I - L - U)x = (I + S_3)b$$
Or, $$(I - L - U + S_3 - S_3L - S_3U)x = (I + S_3)b$$
Or, $$(I - L - U + S_3 - D_1 - L_1 - U_1 - S_3U)x = (I + S_3)b; \ where \ \ S_3L = D_1 + L_1 + U_1$$
Or, $$(I - L - D_1 - L_1)x = (U - S_3 + S_3U + U_1)x + (I + S_3)b$$
Or, $$x = (I - L - D_1 - L_1)^{-1}(U - S_3 + S_3U + U_1)x + (I - L - D_1 - L_1)^{-1}(I + S_3)b$$

Then the proposed method can be expressed as,

$$x^{(k+1)} = T_3x^{(k)} + c_3 \ ;$$

Where, the preconditioned iterative matrix$(T_3) = M_3^{-1}N_3; \ M_3 = I - L - D_1 - L_1$,

$$N_3 = U - S_3 + S_3U + U_1;$$

the preconditioned iterative vector$(c_3) = (I - L - D_1 - L_1)^{-1}(I + S_3)b;$

and the preconditioned system matrix(A_3) $= (I + S_3)A$
$= (I - L - D_1 - L_1) - (U - S_3 + S_3U + U_1)$
$= M_3 - N_3.$

The preconditioned system matrix can also be put as, $A_3 = (I + S_3)A = (a_{ij}^3)$
$= \{a_{ij} - a_{in}a_{nj}\ ;\ 1 \leq i \leq n-1, 1 \leq j \leq n\ a_{nj}\ ;\qquad 1 \leq j \leq n$

Again, it can be easily notice that if,
$a_{in}a_{ni} \neq 1\ ;\ 1 \leq i \leq n-1$, then M_3^{-1} exists and i.e if M_3 is nonsingular, then the preconditioned GS iterative matrix $(T_3) = M_3^{-1}N_3$ is defined. Throughout of this paper, we will assume that $a_{in} \neq 0,\ i = 1,2,\cdots\cdots,n-1.$

Comparison Theorem

Theorem 4.1. Let A be an nxn irreducibly diagonally dominant Z-matrix and $A = M - N$ (GS splitting) & $A_3 = M_3 - N_3$ (preconditioned GS splitting).Then the following inequality holds:

$$M^{-1} \geq 0,\ \ N \geq 0\ \ and\ \ M_3^{-1} \geq 0,\ \ N_3 \geq 0.$$

Also, $A = M - N$ and $A_3 = M_3 - N_3$ are the Gauss-Seidel convergent splittings.

Proof. We have, $I \geq 0,\ \ L \geq 0,\ \ U \geq 0,\ \ D_1 \geq 0,\ \ L_1 \geq 0,\ \ U_1 \geq 0,\ \ S_3 \geq 0.$

Then, $M^{-1} = (I - L)^{-1}$
$= I + L + L^2 + L^3 + \cdots\cdots\cdots\cdots \geq 0;\ \ provided\ \ \rho(L) < 1.$

and $N = U \geq 0.$

Again, $M_3^{-1} = (I - L - D_1 - L_1)^{-1}$
$= [I - (L + D_1 + L_1)]^{-1}$
$= I + (L + D_1 + L_1) + (L + D_1 + L_1)^2 + (L + D_1 + L_1)^3 + \cdots\cdots\cdots \geq 0;$
$provided\ \ \ \rho(L + D_1 + L_1) < 1$

and $N_3 = U - S_3 + S_3U + U_1 \geq 0;\ \ \ since\ \ \ U - S_3 \geq 0.$

i.e. $M^{-1} \geq 0,\ N \geq 0$ and $M_3^{-1} \geq 0,\ N_3 \geq 0$ and so by definition 2.1., $A = M - N$ and $A_3 = M_3 - N_3$ are the Gauss-Seidel convergent splittings. Hence from Theorem 3.29 in [9], we have $\rho\left(M^{-1}N\right) < 1$ and $\rho\left(M_3^{-1}N_3\right) < 1.$

Theorem 4.2. Let A be an nxn irreducibly diagonally dominant Z-matrix and $A = M - N$ is the Gauss-Seidel convergent splitting. Then the Gauss-Seidel iteration matrix $T = M^{-1}N \geq 0.$

Proof. By Theorem 4.1. that,
$$M^{-1} \geq 0,\ \ \ N \geq 0$$
Therefore, $T = M^{-1}N \geq M^{-1} \geq 0.$

Theorem 4.3. Let A be an nxn irreducibly diagonally dominant Z-matrix and $A = M - N, A_3 = M_3 - N_3$ are the Gauss-Seidel convergent splittings. Then

(a) $\rho(T_3) < \rho(T),\ \ \ provided\ \ \rho(T) < 1$
(b) $\rho(T_3) = \rho(T),\ \ \ provided\ \ \rho(T) = 1$
(c) $\rho(T_3) > \rho(T),\ \ \ provided\ \ \rho(T) > 1.$

Proof. By Theorem 4.2., we have $T \geq 0$ i.e. T is a nonnegative square matrix and therefore T has a positive eigenvector say x and $\rho(T) > 0$(Definition 2.9.). Let $x = (x_1, x_2, x_3, \cdots\cdots, x_n)^T$ be the positive eigenvector corresponding to the eigenvalue $\rho(T)$ of T. Then

$$Tx = \rho(T)x$$

Or, $$(I-L)^{-1}Ux = \rho(T)x; \quad since\ \ GS\ iteration\ matrix(T) = (I-L)^{-1}U$$

Or, $$Ux = \rho(T)(I-L)x \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (a)$$

Or, $$[(U - S_3 + S_3U + U_1) + (S_3 - S_3U - U_1)]x = \rho(T)[(I - L - D_1 - L_1) + (D_1 + L_1)]x \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (b)$$

Let $$B = (I - L - D_1 - L_1)^{-1}$$

Now, $$T_3x - Tx = B[(U - S_3 + S_3U + U_1) - \rho(T)(I - L - D_1 - L_1)]x$$

$$= B\big[\rho(T)\big((I - L - D_1 - L_1) + (D_1 + L_1)\big) - (S_3 - S_3U - U_1) - \rho(T)(I - L - D_1 - L_1)x\big],$$

[Using (b)]

$$\begin{aligned}
&= B[\rho(T)(D_1 + L_1) - (S_3 - S_3U - U_1)]x \\
&= B[\rho(T)(S_3L - U_1) - (S_3 - S_3U - U_1)]x \\
&= B[S_3(\rho(T)L - I + U) + (1 - \rho(T))U_1]x \\
&= B[S_3(\rho(T)L - I + \rho(T)(I - L)) + (1 - \rho(T))U_1]x\,; \quad [\ Using\ \ (a)\ ] \\
&= B\big[\big(1 - \rho(T)\big)(U_1 - S_3)\big]x \\
&= \big(1 - \rho(T)\big)B(U_1 - S_3)x \\
&= \big(1 - \rho(T)\big)B(S_3L - D_1 - L_1 - S_3)x \qquad [since\ \ S_3L = D_1 + L_1 + U_1] \\
&= (\rho(T) - 1)B[S_3(I - L) + D_1 + L_1]x \\
&= (\rho(T) - 1)B\left[\frac{1}{\rho(T)}S_3U + D_1 + L_1\right]x \qquad [\ Using\ (a)\ ] \\
&= \frac{(\rho(T)-1)}{\rho(T)}B[S_3U + \rho(T)(D_1 + L_1)]x
\end{aligned}$$

Clearly, $$B[S_3U + \rho(T)(D_1 + L_1)]x \geq 0\,; \quad since\ \ B \geq 0,\ S_3 \geq 0,\ U \geq 0,\ L_1 \geq 0,\ D_1 \geq 0\ \ and\ \ \rho(T) > 0,\ x > 0.$$

Again, $B[S_3U + \rho(T)(D_1 + L_1)]x \neq 0$ and so it is a nonzero, nonnegative vector.
Now, if $\rho(T) < 1$, then obviously $T_3x - Tx \leq 0$. Since T_3 and $T(T_3 \neq T)$ both are nonnegative square matrices and so, $\rho(T_3) < \rho(T)$ [Definition 2.9.]
Similarly, if $\rho(T) = 1$, then clearly $\rho(T_3) = \rho(T)$
and $\rho(T) > 1$ implies $\rho(T_3) > \rho(T)$.

Hence the proposed method converges faster than the standard GS method when $\rho(T) < 1$ and the new method diverges even faster than the standard GS method when $\rho(T) > 1$.

Furthermore, it also has been noted that when the matrix A is large and sparse, then if $\rho(T) < 1$ and approaches to 1, the improvement is very slight. But if $\rho(T) < 1$ and approaches to 0.5 then the proposed method is much preferable over the standard GS method.

Comparison of numerical results
To know the efficiency of the new method we have considered the following three matrices and compared the numerical results for the different GS methods as mentioned in this paper:

(1) $A_1 = (1\ -0.2\ -0.3\ \ -0.2\ -0.2\ -0.1\ \ 1\ -0.2\ \ -0.3\ 1\ \ -0.1\ -0.3\ \ -0.2\ -0.3\ -0.3\ \ 1\ )$

(2) $A_2 = (1\ -0.2\ -0.3\ -0.2\ \ -0.2\ -0.1\ -0.2\ -0.2\ 1\ -0.3\ -0.1\ -0.2\ 1\ -0.3\ \ -0.3\ -0.1\ 1\ \ -0.1\ -0.2\ -0.3\ \ -0.3\ -0.2\ -0.3\ -0.1\ \ 1\ )$

(3) $A_3 = (1\ -0.1\ -0.2\ \ -0.2\ 1\ -0.2\ \ -0.1\ -0.2\ 1\ \ -0.2\ -0.2\ -0.1\ \ -0.1\ -0.1\ -0.2\ \ -0.1\ -0.1\ -0.1\ \ -0.2\ -0.2\ -0.3\ \ -0.1\ -0.2\ -0.1\ \ -0.3\ -0.1\ -0.2\ \ 1\ -0.2\ -0.1\ \ -0.1\ 1\ -0.2\ \ -0.1\ -0.1\ 1\ )$

The spectral radii of the iteration matrices for the different methods as mentioned have been computed using MATLAB R12 and are given in the following table:

A_i	$\rho(T)$	$\rho(T_3)$	$\rho(T_C)$	$\rho(T_{P_1})$	$\rho(T_{P_2})$
A_1	0.4671	0.3144	0.4671	0.4208	0.4596
A_2	0.7081	0.6212	0.7081	0.6702	0.7012
A_3	0.6065	0.5593	0.6065	0.5829	0.6023

Where the symbols $\rho(T)$, $\rho(T_3)$, $\rho(T_C)$, $\rho\left(T_{P_1}\right)$ and $\rho(T_{P_2})$ mean the spectral radii of the iteration matrices of the standard GS method, the proposed method, the preconditioned GS method with the preconditioner $(I + C)$ [6], the preconditioned GS methods when applying the preconditioners $(I + P_1)$ [8] *and* $(I + P_2)$ [8] respectively.

CONCLUSIONS

In this paper we have presented a new preconditioned GS method for the solution of system of linear equations. Some comparison theorems and three numerical examples are added to show the efficiency of the new method. In each case, we have seen that the spectral radius of the proposed method is less than the original GS method, the preconditioned GS method with the preconditioner $(I + C)$ [6] and the preconditioned GS methods with the preconditioners $(I + P_1,\ I + P_2)$ [8]. Hence by Theorem 4.3, it is obvious that the rate of convergence of the proposed method is faster and its error in any level is less than the GS method.

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